

MODELS OF DIFFUSIVE NOISE ON THE SPHERE

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We analyze Haake et al. method [1] for coarse graining quantum maps on the sphere from the point of view of realizable physical quantum operations achieved with completely positive superoperators. We conclude that sharp truncations à la Haake do not fall into this class.

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The dissipative kicked top has been extensively studied by Haake and collaborators [1]. In their work, the Ruelle Pollicot resonances [2] which rule the asymptotic decay of time-dependent quantities are evaluated both classically and quantum mechanically. The coarse graining is implemented by a truncation of the evolution operator (Perron Frobenius or Husimi propagator) to a finite dimension. The procedure consists in introducing a basis of functions ordered by resolution in phase space and then performing a truncation of dimension N . The eigenvalues of the N -dimensional matrix are evaluated and the limit $N \rightarrow \infty$ is finally taken.

An alternative way of implementing dissipation in decoherent quantum maps is via a coarse-graining superoperator. Such a model has been introduced in [3] for chaotic maps on the torus. In this approach the Ruelle resonances are determined as the eigenvalues of a superoperator which is the composition of a unitary and a diffusive part, the latter being represented by a Kraus sum [5].

In this letter the superoperator formalism will be extended to maps on the sphere, more particularly to the kicked top, by appropriately modelling the diffusive step on the sphere. Our purpose is to study the relationship between this approach and the *sharp truncation formalism* of [1]. That is, we will investigate whether a truncation procedure à la Haake can be implemented via a diffusive superoperator, and which specific form such a procedure would imply for it. Reciprocally, it will be interesting to see how a given modelisation of the Kraus superoperator leads to a different criterium for the basis truncation in the *sharp truncation formalism*.

The dynamics of the kicked top is given by an area preserving map, consisting of rotation and torsion operators, acting on a vector $\hat{J} = j (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ of fixed length j , j playing the role of the inverse Planck's constant. The corresponding phase space is the sphere, and $\cos \theta$ and ϕ the canonical variables.

Its quantal version is specified by a Floquet operator:

$$F = R_z(\tau \cos \theta) R_z(\beta_z) R_y(\beta_y) \quad (1)$$

where $R_i(\beta)$ is a rotation around axis i of an angle β . The Hilbert space of the wave functions is spanned by the $(2j+1)$ eigenvectors of \hat{J}_z , $|jm\rangle$. Since we are dealing with open systems states will be represented by a density operator $\hat{\rho}$. In [1] this density operator $\hat{\rho}$ is represented by the corresponding Husimi function $H_\rho(\theta, \phi) = \langle j\theta\phi | \hat{\rho} | j\theta\phi \rangle$, that is, its diagonal matrix element in the basis of coherent states, which in turn is expanded in terms of spherical harmonics:

$$H_\rho(\theta, \phi) = \sum_{L=0}^{2j} \sum_{M=-L}^{M=L} H_\rho(L, M) Y_{LM}(\theta, \phi) \quad (2)$$

The function $H_\rho(L, M)$ can be thought as the representation of $\hat{\rho}$ in an orthogonal basis of operators \hat{T}_{LM} given by $\hat{T}_{LM} = \int d\cos\theta d\phi |j\theta\phi\rangle Y_{LM}^*(\theta, \phi) \langle j\theta\phi|$. These are related to the usual angular momentum states through:

$$\hat{T}_{LM} = (-)^j \sqrt{\frac{4\pi}{(2L+1)}} C_{jjj-j}^{L0} \sum_{m_1, m_2} (-)^{m_1} C_{j-m_1, j, m_2}^{LM} |jm_1\rangle \langle jm_2| \quad (3)$$

where C_{j-m_1, j, m_2}^{LM} are Clebsch Gordan coefficients. The coarse graining is then introduced by fixing a resolution parameter $L_{max} < 2j$ and truncating the Husimi propagator U of matrix elements

$$U_{LM, L'M'} = \text{tr}(\hat{T}_{LM}^\dagger F \hat{T}_{L'M'} F^\dagger) \quad (4)$$

to be diagonalized, to a dimension $(L_{max} + 1)^2$. If the area of the sphere is normalized to 4π then the finest detail of the quantum distribution is the 'sub-planck' cell [4] of size $4\pi/(2j+1)^2$. A truncation of size L_{max} eliminates all structures of size finer than $4\pi/(L_{max}+1)^2$.

We now consider the superoperator formalism. Here the linear action of F on the space of density matrices defines a unitary superoperator $\mathbf{L} = F \otimes F^\dagger$, of dimension $(2j+1)^2 \times (2j+1)^2$. Interaction with the environment and decoherence is then introduced by composing \mathbf{L} with

a diffusion superoperator \mathbf{D}_ϵ , so that the form of the full propagator is : $\mathbf{L}_\epsilon = \mathbf{D}_\epsilon \circ \mathbf{L}$. In analogy with the superoperator defined in [3] for maps on the torus, which was a superposition of translation operators, \mathbf{D}_ϵ is here an incoherent superposition of rotations $R(\Omega)$, each rotation having a probability $c_\epsilon(\Omega)$. Its Kraus representation [5] reads :

$$\mathbf{D}_\epsilon = \int d\Omega c_\epsilon(\Omega) R(\Omega) \otimes R^\dagger(\Omega). \quad (5)$$

and its action on the density matrix will be:

$$\mathbf{D}_\epsilon \rho = \int d\Omega c_\epsilon(\Omega) R(\Omega) \hat{\rho} R^\dagger(\Omega). \quad (6)$$

The preservation of the trace implies that:

$$\begin{aligned} \int d\Omega c_\epsilon(\Omega) &= 1 \\ c_\epsilon(\Omega) &\geq 0. \end{aligned} \quad (7)$$

In addition, an overall drift in any particular direction should be avoided, leading to a further constraint on the form of $c_\epsilon(\Omega)$. If the rotations are expressed as $R(\omega, \Theta, \Phi)$, that is, as rotations through an angle ω around some axis $\hat{n}(\Theta, \Phi)$, isotropy requires that all directions of \hat{n} should be equally probable and thus that the coefficient $c_\epsilon(\Omega)$ should be independent of the direction of \hat{n} and only depend on ω . Furthermore if we take $c_\epsilon(\omega)$ to be a narrow function peaked at $\omega = 0$, the action of \mathbf{D}_ϵ consists in displacing any state incoherently and isotropically over a region of area $\propto \epsilon^2$ on the surface of the sphere. The coarse graining parameter ϵ is then proportional to L_{max}^{-1} , L_{max} being the resolution parameter defined above.

An alternative modelisation of the diffusion superoperator can be given in terms of the operators \hat{T}_{LM} introduced in eq.(3) as:

$$\mathbf{D}'_\epsilon = \sum_{LM} c_\epsilon^L \hat{T}_{LM} \otimes \hat{T}_{LM}^\dagger. \quad (8)$$

In this case the preservation of the trace implies that:

$$\sum_{L=0}^{2j} c_\epsilon^L (C_{jjj-j}^{L0})^2 = \frac{(2j+1)}{4\pi} \quad (9)$$

while the isotropy requirement is satisfied by imposing that the coefficients c_ϵ^L should not depend on the angular momentum projection M . Under this condition one can show that \mathbf{D}'_ϵ coincides with \mathbf{D}_ϵ if the c_ϵ^L are related to the $c_\epsilon(\omega)$ through:

$$c_\epsilon^L = 2(2L+1) \frac{(-)^L}{(C_{jjj-j}^{L0})^2} \sum_K (-)^K \left\{ \begin{matrix} j & j & K \\ j & j & L \end{matrix} \right\} \int d\omega \sin^2 \frac{\omega}{2} c_\epsilon(\omega) \chi^K(\omega) \quad (10)$$

where $\left\{ \begin{matrix} j & j & K \\ j & j & L \end{matrix} \right\}$ indicates the $6j$ symbol and $\chi^K(\omega)$ is the character of the $SU(2)$ irreducible representation of rank K . It is then clear that the diffusion superoperator can be specified indistinctly either by modelling the coefficients $c_\epsilon(\omega)$ or the c_ϵ^L . For example, taking $c_\epsilon(\omega)$ to be a gaussian - like function of width ϵ peaked at $\omega = 0$ corresponds to a c_ϵ^L peaked at $L = 0$ and having a width proportional to $j\epsilon$, as shown in Fig. (1).

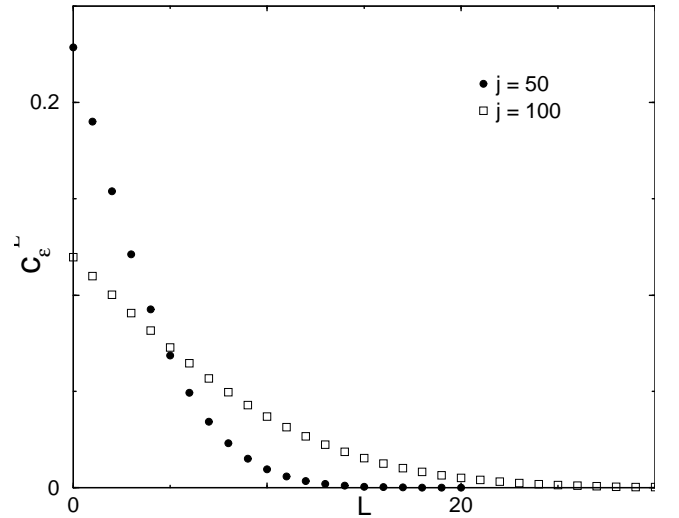


FIG. 1: Normalized coefficients c_ϵ^L corresponding to $c_\epsilon(\omega)$ given by eq.(16) for $\epsilon = 0.1$ and two values of j .

In order to find the link between this model and the *sharp truncation formalism*, the spectral properties of \mathbf{D}_ϵ have to be analyzed. These are not as simple as in the case of maps on the torus, where the eigenfunctions of the translation operators were the translation operators themselves. Here the eigenstates of \mathbf{D}_ϵ depend on the particular form of the coefficients $c_\epsilon(\Omega)$. Nevertheless, it can be shown that, provided \mathbf{D}_ϵ is isotropic, we have:

$$\begin{aligned} (\hat{T}_{L'M'}, \mathbf{D}_\epsilon \hat{T}_{LM}) &= \delta_{LL'} \delta_{MM'} |\hat{T}_{LM}|^2 \frac{1}{(2L+1)} \\ &\int d\omega \sin^2 \frac{\omega}{2} c_\epsilon(\omega) \chi^L(\omega). \end{aligned} \quad (11)$$

That is, the operators \hat{T}_{LM} are eigenstates of \mathbf{D}_ϵ , their corresponding eigenvalues being:

$$e_\epsilon^L = \frac{8\pi}{(2L+1)} \int d\omega \sin^2 \frac{\omega}{2} c_\epsilon(\omega) \chi^L(\omega) \quad (12)$$

independent of M . The fact that \mathbf{D}_ϵ is diagonal in the $\{\hat{T}_{LM}\}$ - basis, which is precisely the one chosen in [1] to achieve the truncation, enables us to write the matrix elements of the full propagator \mathbf{L}_ϵ as:

$$(\mathbf{L}_\epsilon)_{LM,L'M'} = e_\epsilon^L \text{tr}(\hat{T}_{LM}^\dagger F \hat{T}_{L'M'} F^\dagger) \quad (13)$$

and compare this expression with the one corresponding to the matrix elements of the truncated Husimi propagator given in eq.(4). It becomes then clear that the truncation of [1] can be reproduced in our scheme by requiring :

$$e_\epsilon^L = \Theta(L_{max} - L). \quad (14)$$

In this way, the action of the diffusion superoperator \mathbf{D}_ϵ will be to suppress terms with $L > L_{max}$ leaving the others unchanged, or, in other words, to reduce the $(2j+1)^2$ -dimensional \mathbf{L} to the $(L_{max}+1)^2$ -dimensional \mathbf{L}_ϵ . Inverting eq.(12), together with eq.(14) we get the form of the $c_\epsilon(\omega)$ leading to such a spectrum:

$$c_\epsilon(\omega) = \frac{[(2L_{max}+3)\chi^{L_{max}}(\omega) - (2L_{max}+1)\chi^{L_{max}+1}(\omega)]}{32\pi^2 \sin^2 \frac{\omega}{2}}. \quad (15)$$

In Fig. (2), $c_\epsilon(\omega)$ for $L_{max} = 100$ is plotted.

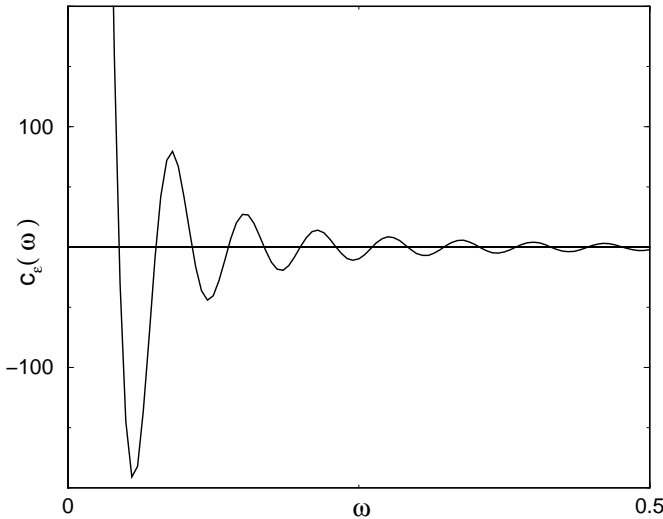


FIG. 2: Coefficient $c_\epsilon(\omega)$ corresponding to the sharp truncation. $L_{max} = 100$.

We see that $c_\epsilon(\omega)$ is a non positive strongly oscillating function of ω . Therefore, we can conclude that a sharp truncation cannot be achieved within the superoperator formalism since it implies coefficients in the Kraus representation which do not fulfill the positivity requirement of eq.(7).

We could also follow the inverse procedure, that is, start from a coefficient $c_\epsilon(\omega)$ which satisfies the superoperator

requirements and use eq.(12) to compute the corresponding eigenvalues of \mathbf{D}_ϵ . For example, taking $c_\epsilon(\omega)$ to be a smooth positive function of ω :

$$c_\epsilon(\omega) = \frac{1}{8\pi \sin^2 \frac{\omega}{2}} \sqrt{\frac{2}{\pi}} \frac{1}{\epsilon} \exp\left[-\frac{\sin^2 \omega}{2\epsilon^2}\right] \quad (16)$$

we get:

$$e_\epsilon^L = \frac{2}{(2L+1)} \left(\frac{1}{2} + \sum_{k=0}^L \exp\left[-\frac{1}{2}k^2\epsilon^2\right] \right) \quad (17)$$

In Fig.(3) these eigenvalues are plotted for $\epsilon = 0.1$. We observe that the action of \mathbf{D}_ϵ is a suppression of the high L components, but now following a smooth function. The width of this function is proportional to $1/\epsilon$ and its rate of decay is small, so that there are still non negligible contributions for $L_{max} \leq L \leq 2j$.

We summarize our results: there are two ways to write down a Kraus form for a diffusive superoperator on the sphere, given by eq.(5) and eq.(8). They both represent the same completely positive superoperator provided the coefficients are related by eq.(10). This superoperator is diagonal in the T_{LM} operator basis and the eigenvalues can be tailored to truncate and coarse-grain unitary maps on the sphere. However sharp truncations result in linear actions that are not interpretable as *physical quantum operations* derived from unitary interactions with an environment - i.e. are not expressed as completely positive superoperators. This is a diffraction effect caused by the sharp edge in L -space. It casts a different light on Haake's group results but of course has no relevance to their calculation of Ruelle resonances, which only depend on the suppression of high frequencies in the quantum distribution.

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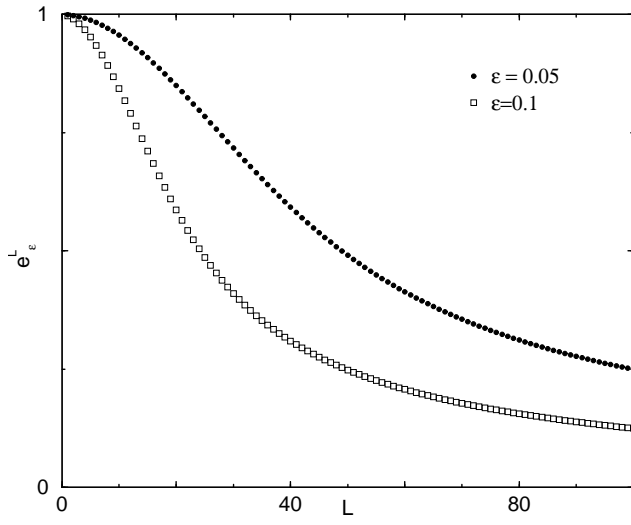


FIG. 3: Eigenvalues of \mathbf{D}_ϵ corresponding to a coefficient $c_\epsilon(\omega)$ given by eq.(16) for $\epsilon = 0.05$ and $\epsilon = 0.1$.